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On the elastic stability of static non-holonomic systems

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Abstract

This paper presents a new formulation for the elastic stability of static non-holonomic structural systems. The theory is developed within the tradition of discrete (or discretized) systems written in terms of a set of generalized coordinates and control parameters. The non-holonomic conditions are written as constraint functions. The formulation employs a Lagrangian functional in terms of the total potential energy, the constraint functions and multipliers. Critical states are identified and the solution is next expanded by regular perturbations. This allows to establish a classification of critical states and identify the initial postcritical behavior. This solution is valid provided that there is no change in the active constraints of the system. The paper presents a mathematical analysis of the critical condition, and concludes with simple examples of two degree-of-freedom systems previously investigated by other authors.

Keywords: Elastic; Stability; Perturbations

1. Introduction

The objective of this paper is to extend the general theory of elastic stability of time independent, discrete structural systems to problems in which there are constraints in the displacement field. Such systems are known as "non-holonomic", and arise in cases in which a new boundary is reached as a consequence of the deformation of the structure (a late boundary condition), and in systems that can only deflect in a given direction (one-way systems).

Problems involving non-holonomic systems that buckle under static loads are common in engineering practice and research. The buckling of perfect and imperfect elastic rings in a cavity under hoop compression was studied by Bucciarelli and Pian (1964), Hsu et al. (1964), Zagustin and Herrmann (1967), and El-Bayoumi (1972). Burgess (1971a) considered imperfect rings in a circular cavity under uniform compression, and showed that there is a critical imperfection amplitude for which buckling may be modeled as a bifurcation. For imperfections smaller than the critical one, the ring recovers the circular shape, while for larger imperfections there is a limit point behavior. Concentric rings or cylinders with different stiffness were

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investigated by Kyriakides (1986). Kyriakides and Youn (1984) concluded that rings in a rigid cavity with a local pressure in the zone enclosed by an inward geometric imperfection display a limit point behavior. This work was extended by Li and Kyriakides (1990, 1991) to investigate buckling propagation in pipelines in a rigid cavity.

An interest in extending the general theory of elastic stability to deal with non-holonomic problems is found, for example, in the works of Burgess (1969, 1971b), Klarbrig (1986), and Godoy (1987).

The basis of the stability theory of structures were laid by the pioneer works of Euler and Lagrange, but what is known at present as the "general theory of elastic stability" under static loads was initiated by Koiter (1945) for continuous systems (see also Como and Grimaldi, 1995). This theory is classified as "general" in the literature partly because of its nonlinearity, but also because it may deal with classes of problems rather than with solutions to specific structural configurations. A recent account of the evolution of the theory may be found in Godoy (2002).

The theory was later formulated for discrete structural systems, and is summarized in a number of books, such as Croll and Walker (1972), Thompson and Hunt (1973), Britvec (1973), Huseyin (1975), El-Naschie (1990), Godoy (2000). An energy formulation is employed in those books and the derivatives of the energy at the critical state are used to produce the general conditions for bifurcation or limit point behavior, and the sensitivity of the critical state. The approach described above employs static bifurcation analysis to investigate instability. Other approaches to buckling attempt to explain the phenomenon using local analysis of bifurcation points, in terms of a pitchfork bifurcation or in terms of saddle-node bifurcation (Wiggins, 1990; Nayfeh and Balachandran, 1995).

Some progress has been obtained to consider non-holonomic systems with theoretical generality. Burgess (1969, 1971b) established conditions to identify a critical state in non-holonomic systems. Klarbrig (1986) considered the problem of equilibrium, uniqueness and stability of nonlinear discrete systems with unilateral contact. Finally, the identification of conditions for regular and critical states was studied by the authors (Mirasso and Godoy, 1992, 1997).

The above contributions have concentrated on the assessment of static stability of an equilibrium state. However, a complete classification of critical states, the relation between an equilibrium path and its stability, and information about the initial postcritical behavior of the system, is still lacking. Those are the specific objectives of this paper.

A summary of previous work in this field carried out by the authors is briefly discussed in Section 2. The classification of critical states is developed in Section 3. Section 4 contains the different possibilities of bifurcation behavior, while the results for limit point behavior are discussed in Section 5. A mathematical analysis of the conditions of critical states is presented in Section 6. Simple link models, previously studied by Burgess (1971a), are solved using the approach presented in this work. Concluding remarks are made in Section 8.

2. Summary of previous work

In a previous paper (Mirasso and Godoy, 1997) the authors formulated the conditions of a critical state in non-holonomic systems by means of constraint functions in terms of the displacements, which are formulated together with the total potential energy of the structure (Burgess, 1971a). The structure is modeled by means of nonlinear kinematic and elastic constitutive equations.

The total potential energy $V(Q_i, \lambda)$ is written in terms of the generalized coordinates Q_i and the load parameter λ. Due to the kinematic nonlinearity the energy becomes quartic in the generalized coordinates and linear in the load parameter. The set of constraints z are grouped in a vector F^Z with components affecting the displacements Q_i, and may be written in the form

$$F^Z(Q_i) = 0 \quad \text{for} \quad Z = 1, \ldots, z \tag{1}$$

The constraints may be active or passive. Active constraints are those for which

$$F^a(Q_i^0) = 0 \quad \text{for} \quad a = 1, \ldots, m \tag{2}$$

in which m is the number of active constraints. Passive constraints satisfy the condition

$$F^p(Q_i^0) < 0 \quad \text{for} \quad p = 1, \ldots, (z - m) \tag{3}$$

where Q_i^0 are the values of the generalized coordinates of a possible configuration of the system. The Lagrangian functional L is defined as

$$L(Q_i, \lambda, \mu_Z) = V(Q_i, \lambda) + \mu_Z F^Z(Q_i) \tag{4}$$

where μ_Z are Lagrange multipliers.

This led to equilibrium conditions in terms of variational inequalities, which were written in the form of Kuhn–Tucker conditions (see, for example, Arora, 1990)

$$\begin{aligned} &\frac{\partial L}{\partial Q_i} = L_i = V_i + \mu_Z F_i^Z|^E = 0 \\ &\mu_Z|^E \geqslant 0, \quad \frac{\partial L}{\partial \mu_Z} = F^Z(Q_i)|^E \leqslant 0 \\ &\mu_Z F^Z(Q_i)|^E = 0 \end{aligned} \tag{5}$$

where the notation $(\ )_i = \frac{\partial(\ )}{\partial Q_i}$ when it is applied on V, F or L; and $(\ )|^E$ indicates that the variables to the left are evaluated at an equilibrium state E. The Einstein convention has been used to indicate summation on the whole range of variables, except on the complementarity condition $\mu_Z F^Z(Q_i)|^E = 0$. An equilibrium state in this problem is provided by a set of displacement coordinates Q_i and load parameter λ that satisfy Eq. (5).

Next, a perturbation analysis is carried out to investigate the uniqueness of the solution and to distinguish between regular and critical states. The variables in Eq. (5) are expanded in the form

$$\begin{aligned} &Q_i(s) = Q_i^E + \dot{q}_i^E s + \tfrac{1}{2}\ddot{q}_i^E s^2 + \cdots \\ &\lambda(s) = \lambda^E + \dot{\lambda}^E s + \tfrac{1}{2}\ddot{\lambda}^E s^2 + \cdots \\ &\mu_Z(s) = \mu_Z^E + \dot{\mu}_Z^E s + \tfrac{1}{2}\ddot{\mu}_Z^E s^2 + \cdots \end{aligned} \tag{6}$$

where $q_i = Q_i - Q_i^E$ is an incremental displacement measured from the equilibrium state; and a dot on top of a variable indicates a derivative with respect to the perturbation parameter s. The perturbation parameter may be a component of the displacement vector or any other scalar quantity that takes a nonzero value along the path being described. The notation of small q_i is used to identify increments in the displacement field (not total displacements), as in Thompson and Hunt (1973).

Substitution of Eq. (6) into (5) leads to the set of perturbation equations of order p

$$\left.\begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}^{(p)} \\ \boldsymbol{\mu}^{(p)} \end{Bmatrix} + \lambda^{(p)} \begin{Bmatrix} \mathbf{l} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{l}^{p-1} \\ \mathbf{f}^{p-1} \end{Bmatrix}\right|^E = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{7}$$

where matrix $\mathbf{L}$ is the Hessian of the Lagrangian L; the columns of matrix $\mathbf{F}$ are the gradient of the active constraints F^a, where the space of active constraints is R^m; $\mathbf{q}^{(p)} \in R^n$, $\lambda^{(p)} \in R$, $\boldsymbol{\mu}^{(p)} \in R^m$ are the unknowns in the perturbation system of order p. For example, in the first-order perturbation system one has $\mathbf{q}^{(p)} = \dot{\mathbf{q}}^E$, $\boldsymbol{\mu}^{(p)} = \dot{\boldsymbol{\mu}}^E$, and $\lambda^{(p)} = \dot{\lambda}^E$. The terms $\mathbf{l}^{p-1}$ and $\mathbf{f}^{p-1}$ are obtained from the solution of the perturbation equation of order $(p - 1)$. Vector $\mathbf{l}$ contains the derivatives of the gradient of the Lagrangian L_i with respect to the control parameter λ and its components may be written as

$$l_i = \frac{\partial L_i}{\partial \lambda} = \frac{\partial (V_i + \mu_Z F_i^Z)}{\partial \lambda} = V_i'$$

Because μ_Z and F_i^Z are not explicit functions of λ, then

$$l_i = \frac{\partial V_i}{\partial \lambda} = V_i' \tag{8}$$

Thus, $\mathbf{l}$ is the vector of generalized loads of the associated holonomic system. The notation $(\ )' = \frac{\partial(\ )}{\partial \lambda}$ has been used in Eq. (8).

For a non-holonomic system, the conditions required to reach a critical state are obtained from Eq. (7), which may also be written as

$$\mathbf{A}\mathbf{d}^{(p)} + \lambda^{(p)} \begin{Bmatrix} \mathbf{l} \\ \mathbf{0} \end{Bmatrix} + \begin{Bmatrix} \mathbf{l}^{p-1} \\ \mathbf{f}^{p-1} \end{Bmatrix} \Bigg|^E = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

where the general matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \tag{9}$$

and the vector of unknowns is

$$\mathbf{d}^{(p)} = \begin{Bmatrix} \mathbf{q}^{(p)} \\ \boldsymbol{\mu}^{(p)} \end{Bmatrix} \tag{10}$$

This matrix $\mathbf{A}$ is associated to the critical states of a non-holonomic system and is analyzed below.

3. Classification of critical states

A system is said to be at a critical state if its matrix $\mathbf{A}$ is singular at a given load level. This means that at a critical state there are vectors $\mathbf{q}^N \in R^n$ and $\boldsymbol{\mu}^N \in R^m$ that are not simultaneously zero, so that

$$\begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{Bmatrix} \Bigg|^C = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{11}$$

or else

$$\mathbf{A}\mathbf{d}^N|^C = \mathbf{0} \quad \text{with} \quad \mathbf{d}^N = \begin{Bmatrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{Bmatrix} \tag{12}$$

The notation $(\ )|^C$ indicates that the variables to the left are evaluated at the critical state.

Vector $\mathbf{d}^N$, or its subvectors, constitute a basis in the kernel of matrix $\mathbf{A}$, i.e. they are the eigenvectors of this matrix associated to a zero eigenvalue. The studies of this paper are restricted to distinct critical states, in the sense that there is only one zero eigenvalue and the dimension of ker($\mathbf{A}$) is 1.

The general solution of Eq. (7) can be written as the sum of particular solutions plus the solution of the homogeneous system (11)

$$\begin{Bmatrix} \mathbf{q}^{(p)} \\ \boldsymbol{\mu}^{(p)} \end{Bmatrix} = \alpha_p \begin{Bmatrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{Bmatrix} + \lambda^{(p)} \begin{Bmatrix} \mathbf{q}^0 \\ \boldsymbol{\mu}^0 \end{Bmatrix} + \begin{Bmatrix} \mathbf{q}^P \\ \boldsymbol{\mu}^P \end{Bmatrix} \Bigg|^C \tag{13}$$

where α_p is a parameter used to scale the eigenvectors d_N, and the two other vectors are obtained from the solution of the following systems:

$$\begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \left\{ \begin{matrix} \mathbf{q}^0 \\ \boldsymbol{\mu}^0 \end{matrix} \right\} \Bigg|^C = - \left\{ \begin{matrix} \mathbf{l} \\ \mathbf{0} \end{matrix} \right\}^C \tag{14}$$

$$\begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \left\{ \begin{matrix} \mathbf{q}^P \\ \boldsymbol{\mu}^P \end{matrix} \right\} \Bigg|^C = - \left\{ \begin{matrix} \mathbf{l}^{p-1} \\ \mathbf{f}^{p-1} \end{matrix} \right\}^C \tag{15}$$

The complete solution of the system is obtained in two steps: First, the independent terms in the system (7) must belong to the image of matrix $\mathbf{A}$. Since the image of $\mathbf{A}$ and the kernel of $\mathbf{A}^{\mathrm{T}}$ belong to complementarity (orthogonal) spaces, then the independent terms of systems (7) must be orthogonal to the basis of the kernel $\mathbf{A}^{\mathrm{T}}$. Since $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, then $\ker(\mathbf{A}^{\mathrm{T}}) = \ker(\mathbf{A})$ and is given by the non-zero solution of the homogeneous system (11). Thus, the following is obtained:

$$\left\langle \left\{ \begin{matrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{matrix} \right\}, \left(\lambda^{(p)} \left\{ \begin{matrix} \mathbf{l} \\ \mathbf{0} \end{matrix} \right\} + \left\{ \begin{matrix} \mathbf{l}^{p-1} \\ \mathbf{f}^{p-1} \end{matrix} \right\} \right) \right\rangle = 0 \tag{16}$$

where $\langle\ ,\ \rangle$ indicates a scalar product. From the above condition, one gets

$$\lambda^{(p)} \langle \mathbf{q}^N, \mathbf{l} \rangle + \langle \mathbf{q}^N, \mathbf{l}^{p-1} \rangle + \langle \boldsymbol{\mu}^N, \mathbf{f}^{p-1} \rangle \big|^C = 0 \tag{17}$$

This is the contraction mechanism identified by Thompson and Hunt (1973) for holonomic systems and allows to obtain $\lambda^{(p)C}$. This contraction mechanism plays the same role as Fredholm's alternative and solvability conditions used by other authors (see Nayfeh, 1981, pp. 456).

Second, the choice of the perturbation parameter leads to a new condition. It is possible to choose any component of $\mathbf{q}$ as the perturbation parameter s. Without loss of generality, it is possible to choose the ith component of $\mathbf{q}$, i.e.

$$s \equiv q_i \tag{18}$$

Then

$$q_i^{(p)} = \delta_{ip} \tag{19}$$

where δ_{ip} is the Kronecker delta function. Considering Eq. (13), one has

$$\delta_{ip} = \alpha_p q_i^N + \lambda^{(p)} q_i^0 + q_i^p \tag{20}$$

The value of the parameter α_p depends on the particular solution chosen for the systems (14), and on the way in which the eigenvector of $\mathbf{A}$ was computed. If one assumes

$$q_i^N = 1 \quad \text{and} \quad q_i^0 = q_i^p = 0 \tag{21}$$

then

$$\alpha_p = \delta_{ip} \tag{22}$$

Other valid choices can be made provided s is taken as an adequate parameter to follow the equilibrium path. If a component q_1 is taken as perturbation parameter, then the eigenvector q^N should not have a zero value in this generalized coordinate.

As seen before, the contraction mechanism is applied to the system (7) and yields Eq. (17). In order to classify the critical states it is necessary to analyze the above Eq. (17).

The coefficient $\langle \mathbf{q}^N, \mathbf{l} \rangle|^C$ is crucial for the classification of critical states. If

$$\langle \mathbf{q}^N, \mathbf{l} \rangle |^C \neq 0 \tag{23}$$

then the state C is a limit point, and from Eq. (17) one can compute

$$\lambda^{(p)C} = -\frac{\langle \mathbf{q}^N, \mathbf{l}^{p-1}\rangle + \langle \boldsymbol{\mu}^N, \mathbf{f}^{p-1}\rangle}{\langle \mathbf{q}^N, \mathbf{l}\rangle}\Bigg|^C \tag{24}$$

On the other hand, cases in which

$$\langle \mathbf{q}^N, \mathbf{l}\rangle|^C = 0 \tag{25}$$

are bifurcation states. These statements will be examined in the next sections of the paper.

Notice that $\mathbf{q}^N$ is the part of the generalized displacements in the critical mode associated to matrix $\mathbf{A}$. Furthermore, vector $\mathbf{l}$ is the generalized load applied to the system.

The conditions (23) and (25) are the projections of the vector of generalized loads, $\mathbf{l}$, in the direction of the critical mode, $\mathbf{q}^N$. This is analogous to the interpretation that is made in the general theory of holonomic systems.

4. Bifurcation states

Let us consider a distinct critical state C, in which the orthogonality condition (25) holds true. Then

$$(q_i^N L_i')|^C = 0 \tag{26}$$

The first system of perturbation equations is given by

$$\begin{bmatrix} L_{ij} & F_i^a \\ F_j^a & 0 \end{bmatrix} \begin{Bmatrix} \dot{q}_j \\ \dot{\mu}_a \end{Bmatrix} + \dot{\lambda}\begin{Bmatrix} L_i' \\ 0 \end{Bmatrix}\Bigg|^C = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{27}$$

where $i, j = 1, n$ and $a = 1, m$. If the contraction mechanism is applied to Eq. (27), then one gets

$$\dot{\lambda}(q_i^N L_i')|^C = 0 \tag{28}$$

This is valid for any value of $\dot{\lambda}$, on account of Eq. (26).

The solution of the first system of perturbation equations can be written in the form

$$\begin{Bmatrix} \dot{q}_j \\ \dot{\mu}_a \end{Bmatrix} = \alpha_1 \begin{Bmatrix} q_j^N \\ \mu_a^N \end{Bmatrix} + \dot{\lambda}\begin{Bmatrix} q_j^0 \\ \mu_a^0 \end{Bmatrix}\Bigg|^C \tag{29}$$

where q_j^0 and μ_a^0 are the particular solutions of Eq. (14).

The second system of perturbation equations is given by

$$\begin{bmatrix} L_{ij} & F_i^a \\ F_j^a & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_j \\ \ddot{\mu}_a \end{Bmatrix} + \ddot{\lambda}\begin{Bmatrix} L_i' \\ 0 \end{Bmatrix} + \begin{Bmatrix} l_j^1 \\ f_a^1 \end{Bmatrix}\Bigg|^C = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

where

$$\begin{aligned} l_i^{1E} &= L_{ijk}\dot{q}_j\dot{q}_k + L_i''\dot{\lambda}^2 + 2(L_{ij}'\dot{q}_j\dot{\lambda} + F_{ij}^Z\dot{q}_j\dot{\mu}_Z) \\ f_a^{1E} &= F_{ij}^Z\dot{q}_i\dot{q}_j \end{aligned} \tag{31}$$

Application of the contraction mechanism and taking Eq. (26) into account yields

$$(q_i^N l_i^1) + (\mu_a^N f_a^1)|^C = 0 \tag{32}$$

so that Eq. (30) has a solution.

The solution of the first system of perturbation, Eq. (29), is substituted into Eq. (32), and yields

$$\mathcal{A}\dot{\lambda}^2 + \mathcal{B}\,\dot{\lambda}\alpha_1 + \mathcal{C}\,\alpha_1^2|^C = 0 \tag{33}$$

where

$$\begin{aligned}
\mathcal{A} &= L_{ijk}q_i^N q_j^0 q_k^0 + L_i'' q_i^N + 2L_{ij}' q_i^N q_j^0 + 2F_{ij}^Z q_i^N q_j^0 \mu_Z^0 + F_{ij}^Z q_i^0 q_j^0 \mu_Z^N \\
\mathcal{B} &= L_{ijk}q_i^N (q_j^N q_k^0 + q_j^0 q_k^N) + 2L_{ij}' q_i^N q_j^N + F_{ij}^Z q_i^N q_j^0 \mu_Z^N + 2F_{ij}^Z q_i^N (q_j^N \mu_Z^0 + q_j^0 \mu_Z^N) + F_{ij}^Z q_i^0 q_j^N \mu_Z^N \\
\mathcal{C} &= L_{ijk}q_i^N q_j^N q_k^N + 3F_{ij}^Z q_i^N q_j^N \mu_Z^N
\end{aligned} \tag{34}$$

This is a quadratic equation in $\dot{\lambda}$, and there are two solutions, so that there are two equilibrium paths as expected in a bifurcation.

4.1. Asymmetric bifurcation

The nature of the bifurcation is given by the coefficient $\mathcal{C}$. If

$$\mathcal{C} \neq 0 \tag{35}$$

the following solutions are obtained from Eq. (33):

$$\dot{\lambda}^C = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}}}{2\mathcal{A}} \tag{36}$$

It was previously assumed that $s = q_1$, so from Eq. (22) one gets $\alpha_1 = 1$. The larger of the roots of $\dot{\lambda}^C$ corresponds to the slope of the fundamental equilibrium path, while the other root is the slope of the secondary equilibrium path.

A first-order perturbation solution has been obtained for the secondary equilibrium path passing through the asymmetric bifurcation, and may now be written as

$$\begin{aligned}
Q_i(q_1) &= Q_i^C + \dot{q}_i^C q_1 + \cdots \\
\lambda(q_1) &= \lambda^C + \dot{\lambda}^C q_1 + \cdots \\
\mu_a(q_1) &= \mu_a^C + \dot{\mu}_a^C q_1 + \cdots
\end{aligned} \tag{37}$$

with

$$\begin{aligned}
\dot{q}_i^C &= q_i^N + \dot{\lambda}^C q_i^0 \\
\dot{\mu}_a^C &= \mu_a^N + \dot{\lambda}^C \mu_a^0
\end{aligned} \tag{38}$$

This solution is valid provided

$$\begin{aligned}
&F^p(Q_i^C) + (F_i^p \dot{q}_i)^C q_1 \leqslant 0 \\
&\mu_a(q_1) \geqslant 0
\end{aligned} \tag{39}$$

are satisfied.

4.2. Symmetric bifurcation

The condition for symmetric bifurcation is

$$\mathcal{C} = 0 \tag{40}$$

in which case the quadratic equation (33) has one zero solution

$$\dot{\lambda}^C = 0 \tag{41}$$

while the other root is the slope of the fundamental path. The solution of the first-order system of perturbation equations is obtained in conjunction with Eqs. (29) and (41), leading to

$$\begin{aligned} \dot{q}_i^C &= \alpha_1 q_1^N \\ \dot{\mu}_a^C &= \alpha_1 \mu_a^N \end{aligned} \tag{42}$$

Eqs. (42) are substituted into Eq. (31) and yield

$$\begin{aligned} l_i^1 &= (L_{ijk} q_k^N q_j^N + 2F_{ij}^a q_j^N \mu_a^N)\alpha_1^2 \\ f_a^1 &= (F_{ij}^a q_i^N q_j^N)\alpha_1^2 \end{aligned} \tag{43}$$

Thus, the general solution of the second system can now be written as

$$\left\{ \begin{matrix} \ddot{q}_i \\ \ddot{\mu}_a \end{matrix} \right\} = \alpha_2 \left\{ \begin{matrix} q_i^N \\ \mu_a^N \end{matrix} \right\} + \ddot{\lambda} \left\{ \begin{matrix} q_i^0 \\ \mu_a^0 \end{matrix} \right\} + \left\{ \begin{matrix} q_i^2 \\ \mu_a^2 \end{matrix} \right\} \Bigg|^C \tag{44}$$

where q_i^2 and μ_a^2 are particular solutions of

$$\begin{bmatrix} L_{ij} & F_i^a \\ F_j^a & 0 \end{bmatrix} \left\{ \begin{matrix} q_i^2 \\ \mu_a^2 \end{matrix} \right\} = - \left\{ \begin{matrix} l_i^1 \\ f_a^1 \end{matrix} \right\} \Bigg|^C \tag{45}$$

where l_i^1 and f_a^1 are given by Eq. (43).

To compute the value of $\ddot{\lambda}$ one must use the contraction mechanism on the third-order perturbation equation, i.e.

$$\begin{bmatrix} L_{ij} & F_i^a \\ F_j^a & 0 \end{bmatrix} \left\{ \begin{matrix} \dddot{q}_i \\ \dddot{\mu}_a \end{matrix} \right\} + \dddot{\lambda} \left\{ \begin{matrix} L_i' \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} l_i^2 \\ f_a^2 \end{matrix} \right\} \Bigg|^C = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \tag{46}$$

where

$$\begin{aligned} l_i^{2E} &= L_{ijk1} \dot{q}_j \dot{q}_k \dot{q}_l + 3[L_{ijk} \dot{q}_j \ddot{q}_k + L_i''' \dot{\lambda}^3] + 3\{L_{ijk}' \dot{q}_j \dot{q}_k \dot{\lambda} + L_{ij}' \ddot{q}_j \dot{\lambda} + L_{ij}'' \dot{q}_j \dot{\lambda}^2 + L_i'' \dot{\lambda}\ddot{\lambda} + L_{ij}' \dot{q}_j \ddot{\lambda} \\ &\quad + F_{ijk}^Z \dot{q}_j \dot{q}_k \dot{\mu}_Z + F_{ij}^Z \ddot{q}_j \dot{\mu}_Z + F_{ij}^Z \dot{q}_j \ddot{\mu}_Z\} \\ f_Z^{2E} &= 3F_{ij}^Z \dot{q}_i \ddot{q}_j + F_{ijk}^Z \dot{q}_i \dot{q}_j \dot{q}_k \end{aligned} \tag{47}$$

This leads to

$$(q_i^N l_i^2) + (\mu_a^N f_a^2)|^C = 0 \tag{48}$$

Next, the solution of the first system of perturbation equations (42) and the second system (44) are introduced into Eq. (47), so that Eq. (48) becomes

$$\mathcal{D}\ddot{\lambda}^C + \mathcal{F}|^C = 0 \tag{49}$$

from which

$$\ddot{\lambda}^C = -\frac{\mathcal{F}}{\mathcal{D}} \tag{50}$$

with

$$\begin{aligned} \mathcal{D} &= \tfrac{3}{2}\mathcal{B}\alpha_1 \\ \mathcal{F} &= \alpha_1^3 [L_{ijkl} q_i^N q_j^N q_k^N q_l^N + 4F_{ijk}^Z q_i^N q_j^N q_k^N \mu_Z^N] + 3\alpha_1 [L_{ijk} q_i^N q_j^N q_k^2 + 2F_{ij}^Z q_i^N q_j^2 \mu_Z^N + F_{ij}^Z q_i^N q_j^N \mu_Z^2] + 3\alpha_1\alpha_2 \mathcal{C} \end{aligned} \tag{51}$$

The second-order perturbation solution is now given by

$$\begin{aligned} Q_i(q_1) &= Q_i^C + \dot{q}_i^N q_1 + \tfrac{1}{2} q_i^2 \ddot{q}_i^C + \cdots \\ \lambda(q_1) &= \lambda^C + \tfrac{1}{2} q_1^2 \ddot{\lambda}^C + \cdots \\ \mu_a(q_1) &= \mu_a^C + \dot{\mu}_a^N q_1 + \tfrac{1}{2} q_1^2 \ddot{\mu}_a^C + \cdots \end{aligned} \tag{52}$$

with

$$\left\{ \begin{array}{c} \ddot{q}_i \\ \ddot{\mu}_a \end{array} \right\} = \ddot{\lambda} \left\{ \begin{array}{c} q_i^0 \\ \mu_a^0 \end{array} \right\} + \left\{ \begin{array}{c} q_i^2 \\ \mu_a^2 \end{array} \right\} \Bigg|^C \tag{53}$$

This solution is valid for

$$\begin{aligned} & F^p(Q_i^C) + (F_i^p \dot{q}_i^N)^C q_1 + (F_i^p \ddot{q}_i + F_{ij}^p \dot{q}_i^N \dot{q}_j^N)^C \frac{q_1^2}{2} \leqslant 0 \\ & \mu_a(q_1) \geqslant 0 \end{aligned} \tag{54}$$

Notice that the relation between the coefficients $\mathcal{D}$ and $\mathcal{B}$ given by Eq. (51) is also present in the theory of holonomic systems.

5. Limit points

A limit point is a critical point in which the generalized load vector L_i' has a non-zero component in the direction of the critical mode q_i^N, i.e.

$$(q_i^N L_i')|^C \neq 0 \tag{55}$$

The contraction of the first system of perturbation equations, together with Eqs. (28) and (55), lead to

$$\dot{\lambda}^C = 0 \tag{56}$$

so that the first system of equations has a solution. The first-order solution is completed with

$$\begin{aligned} \dot{q}_j &= \alpha_1 q_i^N \\ \dot{\mu}_a &= \alpha_1 \mu_a^N \end{aligned} \tag{57}$$

Notice that these values are in coincidence with the first-order solution of the symmetric bifurcation. The independent term of the second-order perturbation condition is given by Eq. (42). The contracted form of the second-order perturbation conditions (Eq. (30)) leads to

$$(q_i^N L_i')\ddot{\lambda}^C + (q_i^N l_i^1) + (\mu_a^N f_a^1)|^C = 0 \tag{58}$$

The value of $\ddot{\lambda}^C$ can be computed from this expression in the form

$$\ddot{\lambda}^C = -\frac{\alpha_1^2 \mathcal{C}}{(q_i^N L_i')} \tag{59}$$

where $\mathcal{C}$ is the coefficient defined in Eq. (34). The second-order solution is completed with Eq. (44), i.e.

$$\left\{ \begin{array}{c} \ddot{q}_i \\ \ddot{\mu}_a \end{array} \right\} = \alpha_2 \left\{ \begin{array}{c} q_i^N \\ \mu_a^N \end{array} \right\} + \ddot{\lambda} \left\{ \begin{array}{c} q_i^0 \\ \mu_a^0 \end{array} \right\} + \left\{ \begin{array}{c} q_i^2 \\ \mu_a^2 \end{array} \right\} \Bigg|^C \tag{60}$$

where (q_i^0, μ_a^0) and (q_i^2, μ_a^2) have already been obtained as the particular solutions of the systems (14) and (45).

The postcritical equilibrium path is now given by

$$\begin{aligned} Q_i(q_1) &= Q_i^C + q_1\dot{q}_i^N + \tfrac{1}{2}q_1^2(\ddot{\lambda}^C q_i^0 + q_i^2) + \cdots \\ \lambda(q_1) &= \lambda^C + \tfrac{1}{2}q_1^2\ddot{\lambda}^C + \cdots \\ \mu_a(q_1) &= \mu_a^C + q_1\dot{\mu}_a^N + \tfrac{1}{2}q_1^2(\ddot{\lambda}^C\mu_a^0 + \mu_a^2) + \cdots \end{aligned} \tag{61}$$

Such expressions are valid as long as the conditions

$$\begin{aligned} &F^p(Q_i^C) + (F_i^p q_i^N)^C q_1 + [F_i^p(\ddot{\lambda}^C q_i^0 + q_i^2) + F_{ij}^p q_i^N q_j^N]^C \frac{q_1^2}{2} \leqslant 0 \\ &\mu_a(q_1) \geqslant 0 \end{aligned} \tag{62}$$

are satisfied. There is a unique postcritical equilibrium path in this problem, with a zero slope at the critical state and a non-zero curvature. All the expressions written here are reduced to those of the general theory for holonomic systems whenever there are no constraints in the displacements, or else they are passive.

6. Mathematical analysis of the critical condition

6.1. On the mathematical meaning of the critical state

At a critical state one is interested in the non-zero solutions of the system (7)

$$\left.\begin{bmatrix} \mathbf{L} & \mathbf{F} \\ \mathbf{F}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{Bmatrix}\right|^C = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{63}$$

where $\mathbf{q}^N \in R^n$, $\boldsymbol{\mu}^N \in R^m$, $\mathbf{L} \in R^{n*n}$, $\mathbf{F} \in R^{n*m}$.

The space normal to the active constraints can be interpreted as the image of the linear transformation $\mathbf{F} : R^m \to R^n$, denoted as $\mathrm{Im}(\mathbf{F})$, with dimension $m < n$. On the other hand, the space tangent to the active constraints is the kernel of $\mathbf{F}^{\mathrm{T}}$, $\ker(\mathbf{F}^{\mathrm{T}})$, that is, it is the kernel of the linear transformation $\mathbf{F}^{\mathrm{T}} : R^n \to R^m$. Furthermore, the columns of the matrix $\mathbf{Z} \in R^{n*(n-m)}$ provide a basis of the tangent space, so that

$$\mathbf{F}^{\mathrm{T}}\mathbf{Z} = \mathbf{0} \tag{64}$$

The tangent space R^{n-m} is related to the space R^n through a linear transformation $\mathbf{Z} : R^{n-m} \to R^n$, with $\ker(\mathbf{Z}) = \{0\}$. It is now possible to claim that the vectors $\mathbf{q}^N$ that are solution of the system (63) satisfy

$$\mathbf{F}^{\mathrm{T}}\mathbf{q}^N = \mathbf{0} \iff \mathbf{q}^N \in \ker(\mathbf{F}^{\mathrm{T}}) \tag{65}$$

$$\mathbf{F}\boldsymbol{\mu}^N = -\mathbf{L}\mathbf{q}^N \iff (\mathbf{L}\mathbf{q}^N) \in \mathrm{Im}(\mathbf{F}) \tag{66}$$

One is interested in those vectors $\mathbf{q}^N$ that belong to the kernel of $\mathbf{F}^{\mathrm{T}}$ and that if the linear transformation $\mathbf{L}$ is applied to them, then they fall into the Image of $\mathbf{F}$. This is illustrated in Fig. 1.

Since $\mathbf{q}^N \in \ker(\mathbf{F}^{\mathrm{T}})$, then one may write a unique linear combination of one basis of the tangent space, i.e. there is a unique vector $\mathbf{t}^N \in R^{(n-m)}$ for which

$$\mathbf{q}^N = \mathbf{Z}\mathbf{t}^N \tag{67}$$

On the other hand, if $\mathbf{L}\mathbf{q}^N$ does not have a projection on the space complementary to $\mathrm{Im}(\mathbf{F})$ (on the tangent space), then $\mathbf{L}\mathbf{q}^N \in \mathrm{Im}(\mathbf{F})$. From this it follows that if

$$\mathbf{Z}^{\mathrm{T}}(\mathbf{L}\mathbf{q}^N) = 0 \tag{68}$$

then it is possible to assure that $\mathbf{L}\mathbf{q}^N \in \mathrm{Im}(\mathbf{F})$.

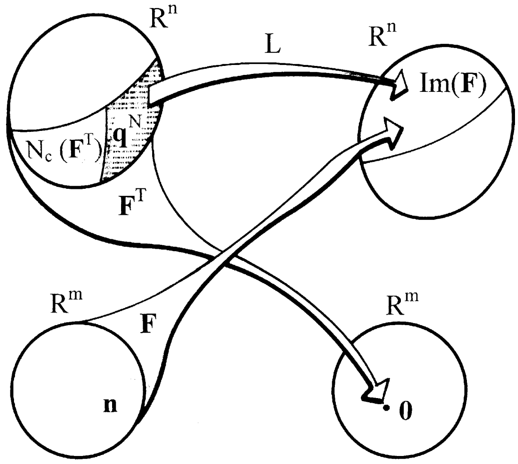

Fig. 1. Critical condition.

The conditions (67) together with (68) lead to

$$(\mathbf{Z}^{\mathrm{T}}\mathbf{L}\mathbf{Z})\mathbf{t}^{N} = 0 \tag{69}$$

or else

$$\mathbf{L}_{p}\mathbf{t}^{N} = 0 \tag{70}$$

From the above it follows that those vectors $\mathbf{q}^{N} \in R^{n}$, which satisfy Eqs. (65) and (66) can be obtained with a linear transformation (67) from $\ker(\mathbf{L}_{p})$. This matrix $\mathbf{L}_{p}$ is the projection of the Hessian of the Lagrangian on the tangent space, and may be seen as a linear transformation from $R^{(n-m)} \to R^{(n-m)}$. This is illustrated in Fig. 2.

The solution of $\ker(\mathbf{A})$ in Eq. (63) can be written as

$$\begin{Bmatrix} \mathbf{q}^{N} \\ \boldsymbol{\mu}^{N} \end{Bmatrix} = \begin{bmatrix} \mathbf{Z} \\ -\mathbf{F}\!\!\!/\,\mathbf{L}\mathbf{Z} \end{bmatrix} \mathbf{t}^{N} \tag{71}$$

where the left inverse of $\mathbf{F}$ has been denoted by matrix $\mathbf{F}\!\!\!/$, and $\mathbf{t}^{N}$ is the critical mode of $\mathbf{L}_{p}$.

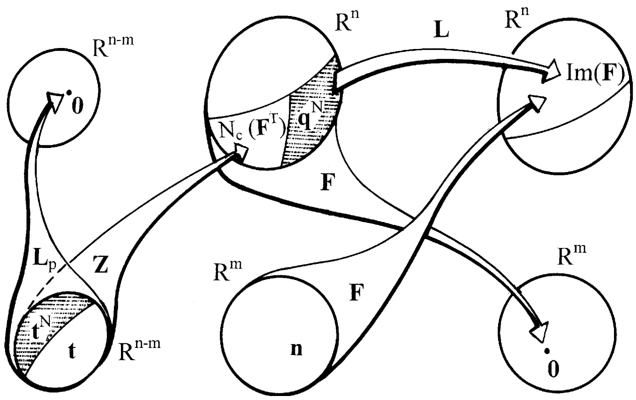

Fig. 2. Components of the critical mode in the tangent space.

The dimension of $\ker(\mathbf{A})$ can vary depending on the transformation $\mathbf{L}: R^n \rightarrow R^n$ and its projection on the tangent space. Some specific cases are investigated in the following sections.

6.2. Positive definite Hessian

There are many situations in which, for a small value of the control parameter λ, then $\mathbf{L}$ is positive definite. This means that the bilinear form $\not{L}(v, v)$

$$\not{L}(\mathbf{v}, \mathbf{v}) \equiv \langle \mathbf{v}, \mathbf{L}\mathbf{v} \rangle = \mathbf{v}^{\mathrm{T}}\mathbf{L}\mathbf{v} > 0 \quad \forall \mathbf{v} \neq 0 \in R^n \tag{72}$$

is always positive for any non-zero vector $\mathbf{v} \in R^n$. Any non-zero vector that belongs to the tangent space can be written in the form

$$\mathbf{v} = \mathbf{Z}\mathbf{t}, \quad \text{if } \mathbf{v} \in \ker(\mathbf{F}^{\mathrm{T}}) \tag{73}$$

where $\mathbf{t} \neq \mathbf{0}$ and $\mathbf{t} \in R^{n-m}$ since the columns of $\mathbf{Z}$ are a basis in the tangent space.

Next, if the vectors of the tangent space defined in Eq. (73) are presented in the bilinear form (72) one gets

$$\begin{aligned} &\not{L}(\mathbf{Z}\mathbf{t}, \mathbf{Z}\mathbf{t}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{Z}\mathbf{L}\mathbf{Z}\mathbf{t} > 0 \quad \forall \mathbf{t} \in R^{n-m} \\ &\not{L}(\mathbf{t}, \mathbf{t}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{L}_p\mathbf{t} > 0 \quad \forall \mathbf{t} \in R^{n-m} \end{aligned} \tag{74}$$

It is possible to conclude that if $\mathbf{L} \in R^{(n*n)}$ is positive definite, its projection on the tangent space, $\mathbf{L}_p$, is also positive definite. Thus, the kernel of $(\mathbf{L}_p)$ is a null vector and this implies that $\mathbf{A}$ is not singular. Thus, any equilibrium state in which $\mathbf{L}$ is positive definite is a normal state (not critical).

6.3. Positive semi-definite Hessian

The Hessian of the Lagrangian is positive semi-definite if

$$\begin{aligned} &\exists \mathbf{x} \neq 0, \quad \mathbf{x} \in R^n : \not{L}(\mathbf{x}, \mathbf{x}) \equiv \mathbf{x}^{\mathrm{T}}\mathbf{L}\mathbf{x} = 0 \\ &\forall \mathbf{v} \neq \mathbf{x}, \quad \mathbf{v} \in R^n : \not{L}(\mathbf{v}, \mathbf{v}) \equiv \mathbf{v}^{\mathrm{T}}\mathbf{L}\mathbf{v} > 0 \end{aligned} \tag{75}$$

for non-zero vectors $\mathbf{x}$ and $\mathbf{v}$.

The directions $\mathbf{x}$ minimize the bilinear form, so they should satisfy the condition

$$\mathbf{L}\mathbf{x} = \mathbf{0} \tag{76}$$

i.e., $\mathbf{x} \in \ker(\mathbf{L})$. It is possible to show that the eigenvalues of $\mathbf{L}$ are non-negative, and that the eigenvectors associated to zero eigenvalues constitute a basis of $\ker(\mathbf{L})$.

There are two cases of interest: first, consider the condition

$$\left\{ \ker(\mathbf{L}) \cap \ker(\mathbf{F}^{\mathrm{T}}) \right\} = \emptyset$$

If for any vector $\mathbf{x} \in \ker(\mathbf{L})$, the condition

$$\mathbf{F}^{\mathrm{T}}\mathbf{x} \neq \mathbf{0} \tag{77}$$

is satisfied, then it is possible to conclude that $\mathbf{x}$ does not belong to the $\ker(\mathbf{F}^{\mathrm{T}})$; i.e. there is no $\mathbf{t} \in R^{n-m}$ that satisfies

$$\mathbf{x} = \mathbf{Z}\mathbf{t} \tag{78}$$

From Eqs. (77) and (75) it follows that for any $\mathbf{t} \in R^{n-m}$ the bilinear form (72) becomes

$$\begin{aligned} &\not{\mathcal{I}}(\mathbf{Zt}, \mathbf{Zt}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{ZLZt} > 0 \quad \forall \mathbf{t} \in R^{n-m} \\ &\not{\mathcal{I}}(\mathbf{t}, \mathbf{t}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{L}_p\mathbf{t} > 0 \quad \forall \mathbf{t} \in R^{n-m} \end{aligned} \tag{79}$$

This situation is similar to what was observed in the previous case, and **A** is non-singular so that the state is a normal point.

Second, consider the condition

$$\{\ker(\mathbf{L}) \cap \ker(\mathbf{F}^{\mathrm{T}})\} \neq \emptyset \tag{80}$$

Any vector $\mathbf{x} \in R^n$, that belongs to the $\ker(\mathbf{L})$, i.e.

$$\mathbf{Lx} = \mathbf{0} \tag{81}$$

and at the same time $\mathbf{x} \in \ker(\mathbf{F}^{\mathrm{T}})$

$$\mathbf{F}^{\mathrm{T}}\mathbf{x} = \mathbf{0} \tag{82}$$

leads to the following solution of Eq. (63):

$$\begin{Bmatrix} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{Bmatrix} = \begin{Bmatrix} \mathbf{x} \\ \mathbf{0} \end{Bmatrix} \tag{83}$$

Thus, matrix **A** is singular and the minimum dimension of $\ker(\mathbf{A})$ is the dimension arising from the intersection between $\ker(\mathbf{L})$ and $\ker(\mathbf{F}^{\mathrm{T}})$. But this is only a condition of minimum dimension, since it is possible to obtain solutions of Eq. (71) that are not coincident with (83) because $\mathbf{L}_p$ becomes singular. In fact, whenever the conditions (81) and (82) are satisfied, it is possible to say that there are vectors $\mathbf{t}^* \in R^{n-m}$ for which

$$\mathbf{x} = \mathbf{Zt}^* \tag{84}$$

and which generate a subspace in R^{n-m}.

From Eq. (75) the following conclusions are obtained:

- For all $\mathbf{t}^0$ that belongs to the subspace of R^{n-m} generated by $\mathbf{t}^*$ according to Eq. (84), then

$$\begin{aligned} &\not{\mathcal{I}}(\mathbf{Zt}^0, \mathbf{Zt}^0) \equiv \mathbf{t}^{0\mathrm{T}}\mathbf{ZLZt}^0 = 0 \\ &\not{\mathcal{I}}(\mathbf{t}^0, \mathbf{t}^0) \equiv \mathbf{t}^{0\mathrm{T}}\mathbf{L}_p\mathbf{t}^0 = 0 \end{aligned} \tag{85}$$

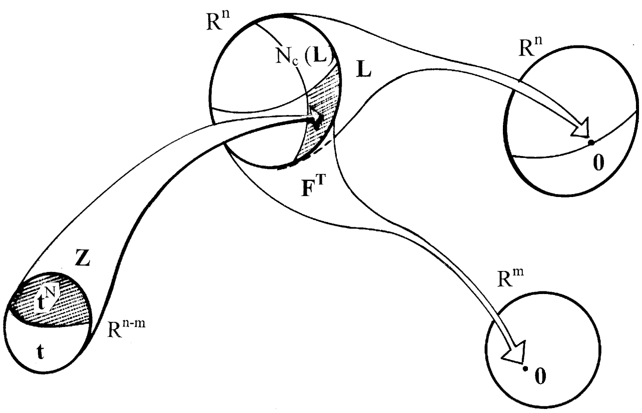

Fig. 3. The singularity condition.

- For $\mathbf{t}$ that does not belong to the subspace of R^{n-m} generated by $\mathbf{t}^*$ according to Eq. (84), then

$$\begin{aligned} &\not{V}(\mathbf{Zt},\mathbf{Zt}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{ZLZt} > 0 \\ &\not{V}(\mathbf{t},\mathbf{t}) \equiv \mathbf{t}^{\mathrm{T}}\mathbf{L}_p\mathbf{t} > 0 \end{aligned} \tag{86}$$

so that $\mathbf{L}_p$ is positive semi-definite.

The conditions that arise when $\mathbf{L}$ is singular are shown in Fig. 3.

7. Illustrative examples

To better understand the developments presented in this paper, two link models, originally studied by Burgess (1971a), are considered in this section.

7.1. System with two constraints

The first model is shown in Fig. 4 together with the notation employed in the analysis. The model has pin joints which allow rotation of the links in the plane of the drawing. In the unloaded state the springs are free of forces and the displacements at B and C are given by $Q_1 = d/2$ and $Q_2 = d/4$.

There are two constraints located at a distance d from line AD, with $d \ll L$. The constraints of this problem may be written in terms of two linear inequalities:

$$\begin{aligned} F^1 &= -d + Q_1 \leqslant 0 \\ F^2 &= -d + Q_2 \leqslant 0 \end{aligned}$$

The total potential energy of the structural system results in

$$V = \frac{k}{2L^2}\left[5Q_1^2 - 8Q_1Q_2 + 5Q_2^2 - \frac{3}{2}Q_2 d - 3Q_1 d + \frac{9}{16}d^2 - \lambda\left(2Q_1^2 - 2Q_1Q_2 + 2Q_2^2 - \frac{3}{8}d^2\right)\right] + \cdots$$

For the system without constraints (in which case both F^1 and F^2 are negative) the equilibrium conditions are $V_i^E = 0$, leading to the solution of the associated holonomic system

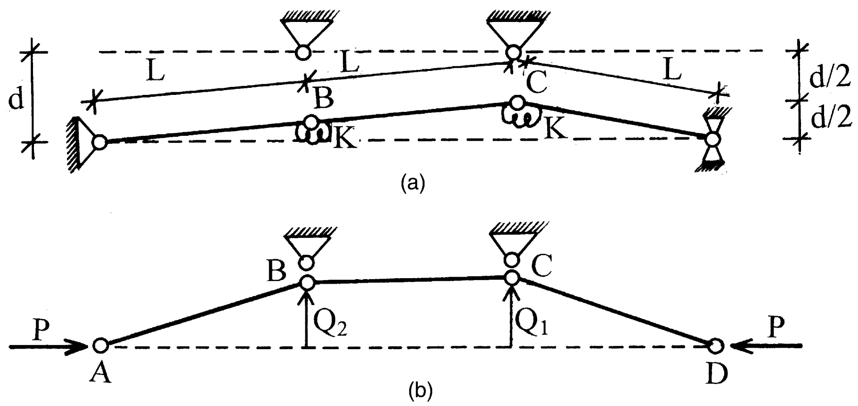

Fig. 4. Two degree of freedom link model with two constraints: (a) unloaded system and (b) loaded configuration.

$$Q_1 = -\frac{3}{4}d\frac{(\lambda-2)}{(\lambda-3)(\lambda-1)} \quad Q_2 = \frac{3}{4}d\frac{1}{(\lambda-3)(\lambda-1)}$$

For stability both the determinant $|V_{ij}|$ and the principal minor V_{11} should be positive, and this is satisfied only for values of $\lambda < 1$. This solution has three equilibrium path, of which one is stable and two are unstable.

(a) Before stability is lost, Q_1 reaches its limit value $Q_1 = d$ at $\lambda = 0.56$ and $Q_2 = 0.69d$. For values of $\lambda > 0.56$ the systems has active constraints and the present theory needs to be applied. For the system with one active constraint, $\lambda \geqslant \lambda_1 = 0.56$, the equilibrium conditions are

$$\frac{k}{L^2}\begin{bmatrix}(5-2\lambda) & (-4+\lambda)\\(-4+\lambda) & (5-2\lambda)\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\end{Bmatrix} + d\frac{k}{L^2}\begin{Bmatrix}-\frac{3}{2}\\+\frac{3}{4}\end{Bmatrix} + \mu_1\begin{Bmatrix}1\\0\end{Bmatrix} = 0$$

$$F^1 = -d + Q_1 = 0 \quad \mu_1 \geqslant 0$$

The solution is given by

$$Q_2 = -d\frac{\left(\lambda - \frac{13}{4}\right)}{(5-2\lambda)} \quad \mu_1 = -d\frac{k}{L^2}\frac{3(\lambda-\lambda_1)(\lambda-\lambda_2)}{(5-2\lambda)}$$

where $\lambda_1 = 0.56$ and $\lambda_2 = 2.70$. Notice that the Lagrange multiplier is non-zero only in the range $\lambda_1 \leqslant \lambda \leqslant 10/4$.

(b) For $\lambda = 7/4$, then $Q_2 = d$ and $\mu_1 = (9/4)kL^2d$, and there are two active constraints. The equilibrium state for which both constraints are active for the first time is defined by

$$Q_1 = Q_2 = d \quad \lambda = \frac{7}{4} \quad \mu_1 = \frac{9}{4}d\frac{k}{L^2}$$

The Kuhn–Tucker conditions become

$$\frac{k}{L^2}\begin{bmatrix}(5-2\lambda) & (-4+\lambda)\\(-4+\lambda) & (5-2\lambda)\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\end{Bmatrix} + d\frac{k}{L^2}\begin{Bmatrix}-\frac{3}{2}\\\frac{3}{4}\end{Bmatrix} + \mu_1\begin{Bmatrix}1\\0\end{Bmatrix} + \mu_2\begin{Bmatrix}0\\1\end{Bmatrix} = 0$$

$$F^1 = Q_1 - d = 0 \quad F^2 = Q_2 - d = 0$$

$$\mu_1 = d\frac{k}{L^2}\left(\lambda + \frac{1}{2}\right) \geqslant 0 \quad \mu_2 = d\frac{k}{L^2}\left(\lambda + \frac{7}{4}\right) \geqslant 0$$

Burgess (1971a) obtained analytical results for this problem; however, a perturbation solution is discussed in this section to illustrate the formulation developed. The solutions are obtained for equilibrium paths emerging from the following three states E_1 (at $\lambda_1 = 0.56$), E_C (at $\lambda_1 = 1$) and E_2 (at $\lambda_1 = 7/4$).

7.1.1. *Analysis of the equilibrium state E_1*

Let E_1 be the state given by

$$\lambda_1 = 0.56 \quad Q_1^E = Q_2^E = d \quad \mu_1^E = 0$$
$$F^1 = 0 \quad F^2 < 0$$

The first-order perturbation system is

$$\mathbf{L} \equiv \mathbf{V} = \frac{k}{L^2}\begin{bmatrix} 3.886 & -3.443 \\ -3.443 & 3.886 \end{bmatrix}$$

$$\mathbf{l} = d\frac{k}{L^2}\begin{Bmatrix} -1.31 \\ -0.38 \end{Bmatrix} \quad \mathbf{F}^{\mathrm{T}} = \{\,1 \quad 0\,\}$$

The computations specified in Section 3 are carried out at the equilibrium states E_1 and yield the perturbation approximation up to the second-order

$$Q = Q^{E1} + s\dot{q}^{E1} + \tfrac{1}{2}s^2\ddot{q}^{E1}$$
$$\mu_1 = \mu_1^{E1} + s\dot{\mu}_1^{E1} + \tfrac{1}{2}s^2\ddot{\mu}_1^{E1}$$
$$\lambda_1 = \lambda^{E1} + s\dot{\lambda}_1^{E1} + \tfrac{1}{2}s^2\ddot{\lambda}^{E1}$$

with

$$\dot{q} = d\dot{\lambda}\begin{Bmatrix} 0 \\ +0.09779 \end{Bmatrix}, \quad \ddot{q} = d\ddot{\lambda}\begin{Bmatrix} 0 \\ 0.09779 \end{Bmatrix} + d\dot{\lambda}^2\begin{Bmatrix} 0 \\ 0.1006554 \end{Bmatrix} \quad \dot{\mu}_1 = +1.64668\frac{k}{L^2}d\dot{\lambda},$$
$$\ddot{\mu}_1 = \frac{k}{L^2}d(1.64669\ddot{\lambda} + 0.150983\dot{\lambda}^2)$$

7.1.2. Analysis of the equilibrium state E_C

Let E_C be the equilibrium state given by

$$\lambda_1 = 1 \quad Q_1^E = d, \quad Q_2^E = \tfrac{3}{4}, \quad \mu_1^E = \tfrac{3}{4}kL^2d \quad F^1 = 0, \quad F^2 < 0$$

In a related system without any constraints, this state would be classified as critical.

The first-order perturbation system takes the form

$$\mathbf{L} \equiv \mathbf{V} = \frac{k}{L^2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

The matrix of the coefficients of $\mathbf{L}$ is singular, and the critical mode associated to the eigenvalue is $\mathbf{x}^{\mathrm{T}} = \{1, 1\}$. Next, one must investigate if this eigenvector belongs to the tangent space with active constraints, using the condition included in Eq. (7)

$$\mathbf{F}^{\mathrm{T}}\mathbf{x} = \{\,1 \quad 0\,\}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 1 \neq 0$$

Because the result is not zero, then one gets the surprising result that $\mathbf{x}$ does not belong to the tangent space and as such the state is not critical but it is a normal point.

The following computations are next obtained to evaluate the unknowns in the perturbation approximation:

$$\mathbf{F}^{\mathrm{T}} = \{1 \quad 0\} \quad \mathbf{F}^{\mathrm{T}}\mathbf{Z} = 0 \rightarrow \mathbf{Z} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\mathbf{L}_p = \mathbf{Z}^{\mathrm{T}}\mathbf{L}\mathbf{Z} \rightarrow L_p = \frac{k}{L^2}3$$

$$l = d\frac{k}{L^2}\begin{Bmatrix} -\frac{3}{4} \\ -\frac{1}{2} \end{Bmatrix}, \quad l_p = Z^{\mathrm{T}} l \rightarrow l_p = -\left(\frac{1}{2}\right)\frac{k}{L^2}d$$

$$L_p t^o = -l_p \rightarrow t^o = \frac{1}{6}d$$

$$\dot{t} = t^o\dot{\lambda} \rightarrow \dot{t} = \left(\frac{1}{6}\right)\dot{\lambda}d$$

$$\dot{q} = Z\dot{t} \rightarrow \dot{q} = d\dot{\lambda}\begin{Bmatrix} 0 \\ \left(\frac{1}{6}\right) \end{Bmatrix}$$

$$a = L\dot{q} + l\dot{\lambda} \rightarrow a = \frac{k}{L^2}d\dot{\lambda}\begin{Bmatrix} -\left(\frac{1}{2}\right) \\ +\left(\frac{1}{2}\right) \end{Bmatrix} + \frac{k}{L^2}d\dot{\lambda}\begin{Bmatrix} -\left(\frac{5}{4}\right) \\ -\left(\frac{1}{2}\right) \end{Bmatrix}$$

$$F\dot{\mu} = -a \rightarrow \dot{\mu}_1 = +\left(\frac{7}{4}\right)\frac{k}{L^2}d\dot{\lambda}$$

The second-order perturbation system requires

$$l^1 = \frac{k}{L^2}d\dot{\lambda}^2\begin{Bmatrix} +\left(\frac{1}{6}\right) \\ -\left(\frac{1}{3}\right) \end{Bmatrix}$$

$$f^1 = 0$$

Under the present formulation the solution of the intermediate equations are

$$F^{\mathrm{T}}F\ddot{n} = -f^1 \rightarrow \ddot{n} = 0$$

$$l_p^1 = Z^{\mathrm{T}}l^1 \rightarrow l_p^1 = -\left(\frac{1}{3}\right)\frac{k}{L^2}d\dot{\lambda}^2$$

$$L_p t_1^2 = -l_p^1 \rightarrow t_1^2 = \left(\frac{1}{9}\right)d\dot{\lambda}^2$$

$$\ddot{t} = t^o\ddot{\lambda} + t_1^2 \rightarrow \ddot{t} = \left(\frac{1}{6}\right)\ddot{\lambda}d + \left(\frac{1}{9}\right)d\dot{\lambda}^2$$

$$\ddot{q} = Z\ddot{t} + F\ddot{n} \rightarrow \ddot{q} = d\ddot{\lambda}\begin{Bmatrix} 1 \\ \frac{1}{6} \end{Bmatrix} + d\dot{\lambda}^2\begin{Bmatrix} 0 \\ \frac{1}{9} \end{Bmatrix}$$

$$a = L\ddot{q} + l\ddot{\lambda} + l^1 \rightarrow a = \frac{k}{L^2}d\begin{Bmatrix} \left(-\frac{1}{2}-\frac{5}{4}\right)\ddot{\lambda} + \left(-\frac{1}{3}+\frac{1}{6}\right)\dot{\lambda}^2 \\ \left(+\frac{1}{2}-\frac{1}{2}\right)\ddot{\lambda} + \left(+\frac{1}{3}-\frac{1}{3}\right)\dot{\lambda}^2 \end{Bmatrix} \quad F\ddot{\mu} = -a \rightarrow \ddot{\mu}_1 = \frac{k}{L^2}d\left(\frac{7}{4}\ddot{\lambda} + \frac{1}{6}\dot{\lambda}^2\right)$$

With the previous results, the second-order perturbation approximation emerging from E_C may be written as

$$Q = Q^{E_C} + s\dot{q}^{E_C} + \tfrac{1}{2}s^2\ddot{q}^{E_C}$$
$$\mu_1 = \mu_1^{E_C} + s\dot{\mu}_1^{E_C} + \tfrac{1}{2}s^2\ddot{\mu}_1^{E_C}$$
$$\lambda_1 = \lambda^{E_C} + s\dot{\lambda}^{E_C} + \tfrac{1}{2}s^2\ddot{\lambda}^{E_C}$$

The solution shows that the equilibrium state E_C is a critical state of the associated holonomic system, and is a singular point of the Hessian of the Lagrangian; however, the non-holonomic system shows a behavior that is characterized as a normal point. This is due to the fact that the critical mode of the holonomic system (in this case it is the critical model of the Hessian of the Lagrangian) has a non-zero projection on the gradient of the active constraints. Thus the constraints have the effect of stiffening the system.

7.1.3. Analysis of the equilibrium state E_2

Finally, let E_2 be the equilibrium state given by

$$\lambda_1 = \tfrac{7}{4} \quad Q_1^E = d \quad Q_2^E = d, \quad \mu_1^E = \tfrac{9}{4}kL^2 d \quad \mu_2^E = 0, \quad F^1 = F^2 = 0$$

The first-order perturbation system is given by

$$\mathbf{L} \equiv \mathbf{V} = \frac{k}{L^2}\begin{bmatrix} \frac{3}{2} & -\frac{9}{4} \\ -\frac{9}{4} & \frac{3}{2} \end{bmatrix} \quad \mathbf{l} = d\frac{k}{L^2}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \quad F^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution of the first-order system is

$$\mathbf{F}^{\mathrm{T}}\mathbf{Z} = 0 \to \mathbf{Z} = 0 \quad \dot{q} = 0 \quad \dot{\mu} = d\frac{k}{L^2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\dot{\lambda}$$

For the second-order system one gets

$$\ddot{q} = 0 \quad \ddot{\mu} = d\frac{k}{L^2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\ddot{\lambda}$$

With the previous results, the second-order perturbation approximation emerging from E_2 is

$$Q = Q^{E_2} + s\dot{q}^{E_2} + \tfrac{1}{2}s^2\ddot{q}^{E_2} \quad \mu_1 = \mu_1^{E_2} + s\dot{\mu}_1^{E_2} + \tfrac{1}{2}s^2\ddot{\mu}_1^{E_2} \quad \lambda = \lambda^{E_2} + s\dot{\lambda}^{E_2} + \tfrac{1}{2}s^2\ddot{\lambda}^{E_2}$$

The equilibrium paths of this example are plotted in Fig. 5, which is in agreement with Fig. 3 of the paper by Burgess (1971a). The solutions obtained from the system of perturbation equations are consistent with the observation that the only two degrees of freedom of the system have constraints in the final state analyzed. Thus, there is only one evolution of the Lagrangian multipliers associated to the support conditions.

7.2. System with one constraint

A second example considered by Burgess (1971a) is studied in this section under the present formulation. The system, shown in Fig. 6, is formed by rigid links joined by springs. The kinematic relation between the degrees of freedom Q_2 and Q_1 and the deformation of the springs is assumed as nonlinear. There is one control parameter $\lambda = PL/k$, where P is the compression along the axis AD, and k is the elastic constant of

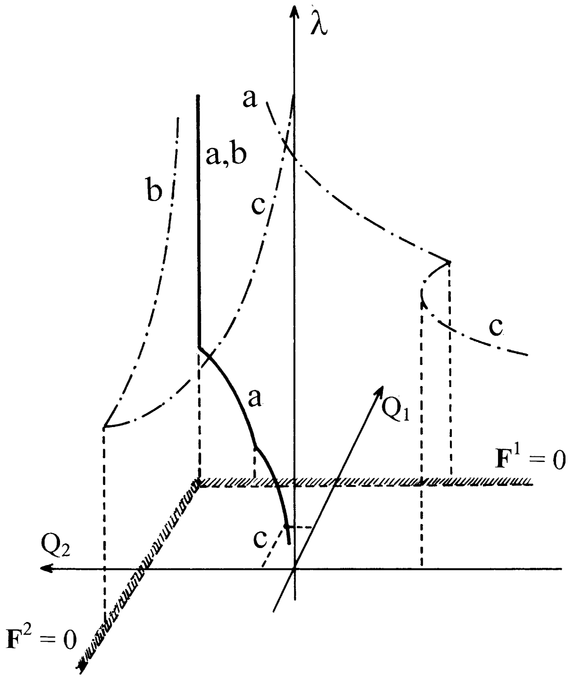

Fig. 5. Equilibrium paths for the model of Fig. 4. Key: (—) stable path; (- - -) unstable path.

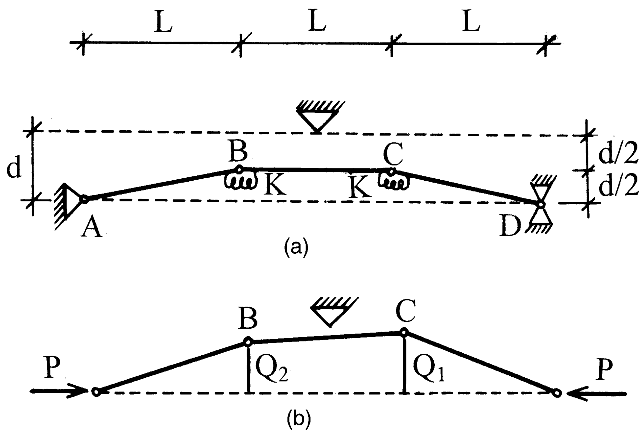

Fig. 6. Two degree of freedom link model with one constraint: (a) unloaded system and (b) loaded configuration.

the linear torsional springs. In the initial configuration, the position of the joints is given by $Q_2 = Q_1 = (1/2)d$ and the springs do not have initial loads.

The total potential energy becomes

$$V = \frac{k}{2L^2}\left[5Q_1^2 - 8Q_1Q_2 + 5Q_2^2 - d(Q_2 + Q_1) + \frac{1}{2}d^2 - \lambda\left(2Q_1^2 - 2Q_1Q_2 + 2Q_2^2 - \frac{1}{2}d^2\right)\right]$$

There is only one constraint in this problem, given by

$$F^1 = -2d + Q_1 + Q_2 \leqslant 0$$

Before the constraint F^1 becomes active, the system behaves as holonomic and the equilibrium path is governed by

$$Q_2 = Q_1 = \frac{d}{2}\frac{1}{1-\lambda}$$

Along the initial path the solution is valid for $0 \leqslant \lambda \leqslant 0.5$. Critical states of the holonomic system are given by

$$\begin{bmatrix} 5-2\lambda & -4+\lambda \\ -4+\lambda & 5-2\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and yield two distinct critical states $\lambda^{C_1} = 1$ with $x_1 = 1$, $x_2 = 1$; and $\lambda^{C_2} = 3$ with $x_1 = 1$, $x_2 = -1$.

For $\lambda = 0.5$ the constraint $F^1 = 0$ becomes active. A plot of the equilibrium paths in this problem is shown in Fig. 7.

7.2.1. Analysis of the equilibrium state C_1

The parameters that define the critical state of the associated holonomic system C_1 are

$$\lambda^{C_1} = 1, \quad Q_1^{C_1} = Q_2^{C_1} = d, \quad \mu_1^{C_1} = \frac{1}{2}d\frac{k}{L^2}, \quad F^1 = 0$$

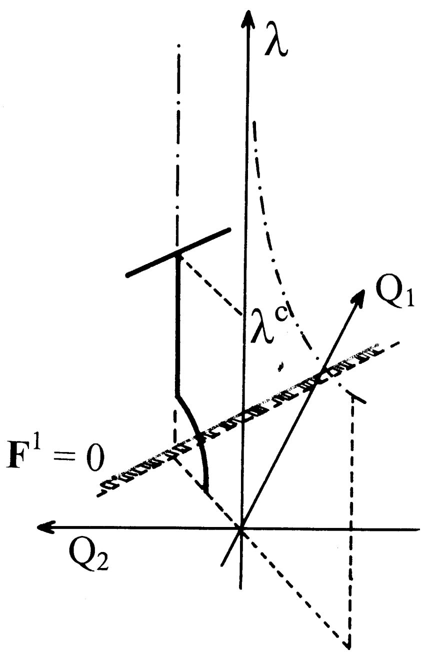

Fig. 7. Equilibrium paths for the model of Fig. 6. Key: (—) stable path and (- - -) unstable path.

The Hessian of the Lagrangian at this state is

$$\mathbf{L} \equiv \mathbf{V} = \frac{k}{L^2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

Because in this problem the constraints are linear in the displacements, then the singularity condition of the Hessian of the Lagrangian L_{ij} are the same as the singularity of the the energy V_{ij} for the associated holonomic system. Thus, matrix **A** of the perturbation systems may become singular depending on the critical mode of the associated holonomic system.

The active constraints define the following normal and tangent spaces

$$\mathbf{F}^{\mathrm{T}} = \{1 \quad 1\} \quad \mathbf{F}^{\mathrm{T}}\mathbf{Z} = 0 \rightarrow \mathbf{Z} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

The critical mode of the associated holonomic system does not belong to the tangent space because the following dot product is not zero:

$$\mathbf{F}^{\mathrm{T}}\mathbf{x}^{C_1} = 2 \neq 0$$

The projection of the Hessian of the Lagrangian on the tangent space is

$$\mathbf{L}_p = \mathbf{Z}^{\mathrm{T}}\mathbf{L}\mathbf{Z} \rightarrow \mathbf{L}_p = 12\frac{k}{L^2}$$

Since this $\mathbf{L}_p$ is positive definite, it is possible to say that matrix **A** of the perturbation system is non-singular, and thus the critical state of the associated holonomic system is a normal point of the non-holonomic system.

To compute the perturbation approximation the load vector is

$$\mathbf{l} = d\frac{k}{L^2}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix}$$

The projection on the tangent space is

$$l_p = \mathbf{Z}^{\mathrm{T}}\mathbf{l} = 0$$

Finally, the solution of the perturbation analysis yields

$$Q_1 = Q_2 = d$$

$$\lambda = 1 + s\dot{\lambda}^{C_1} + \frac{1}{2}s^2\ddot{\lambda}^{C_1}$$

$$\mu = \frac{1}{2}d\frac{k}{L^2} + s\dot{\mu}^{C_1} + \frac{1}{2}s^2\ddot{\mu}^{C_1}$$

where

$$\dot{\mu}^{C_1} = d\frac{k}{L^2}\dot{\lambda}^{C_1} \quad \text{and} \quad \ddot{\mu}^{C_1} = d\frac{k}{L^2}\ddot{\lambda}^{C_1}$$

7.2.2. Analysis of the equilibrium state C_2

The parameters that define the critical state of the associated holonomic system C_2 are

$$\lambda^{C_2} = 3 \quad Q_1^{C_2} = Q_2^{C_2} = d \quad \mu_1^{C_2} = \frac{5}{2}d\frac{k}{L^2} \quad F^1 = 0$$

The Hessian of the Lagrangian at this state is

$$\mathbf{L} \equiv \mathbf{V} = \frac{k}{L^2}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

The active constraints define the following normal and tangent spaces

$$\mathbf{F}^{\mathrm{T}} = \{\, 1 \quad 1 \,\} \quad \mathbf{F}^{\mathrm{T}}\mathbf{Z} = 0 \rightarrow \mathbf{Z} = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$$

The critical mode of the associated holonomic system belongs to the tangent space because the following dot product is zero:

$$\mathbf{F}^{\mathrm{T}}\mathbf{x}^{C_2} = 0$$

Matrix **A** of the perturbation system is singular and the critical state of the associated holonomic system is also a critical state in the non-holonomic system. The critical mode of the non-holonomic system is

$$\left\{ \begin{array}{c} \mathbf{q}^N \\ \boldsymbol{\mu}^N \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{x} \\ \mathbf{0} \end{array} \right\}$$

To compute the perturbation approximation the load vector is

$$\mathbf{l} = d\frac{k}{L^2}\left\{ \begin{array}{c} -1 \\ -1 \end{array} \right\}$$

To classify the critical state, one has to look at the dot product of the critical mode of the non-holonomic system and the load vector of the perturbation system. This is

$$\langle \mathbf{q}^N, \mathbf{l} \rangle = 0$$

Because the load vector is orthogonal to the critical mode, then this critical state is a bifurcation in the non-holonomic system.

8. Conclusions

The classification of distinct critical states of equilibrium has been presented in this paper for non-holonomic structural systems. The classification takes into account the equilibrium paths that pass through such states. The procedure employs a regular perturbation expansion at the critical state in order to find an approximation to the secondary equilibrium path emerging from the critical state. This path is valid as long as the conditions of constraints do not change in the system; this is a limitation for the range of validity of the secondary path, but not for the classification of the critical point itself.

The study presented the conditions required for limit and bifurcation states. The conditions are similar to what is obtained in holonomic systems, and involve the generalized load term and a subvector associated to the displacements in the rigid mode of the matrix of coefficients of the systems of perturbation equations.

It was also possible to isolate the conditions required for different types of bifurcation (asymmetric and symmetric with positive and negative curvature of the emerging path). If the constrains are passive, then the expressions reduce to their counterpart in the holonomic system (Flores and Godoy, 1992; Godoy, 2000).

One of the features of the formulation is that the conditions of critical state have been related to the properties of the Hessian of the Lagrangian. The study led to the form of the null space of the matrix of the coefficients of the perturbation systems by employing the null spaces of the Hessian of the Lagrangian and its projection on the tangent space to the active constraints. A detailed analysis of the bilinear forms associated to the Hessian of the Lagrangian is also presented. The connections between the properties of the Hessian of the Lagrangian and the type of bifurcation point may also be explained as when normal forms are obtained in local bifurcation analysis.

From this research, it follows that the form of the equilibrium paths that pass through a critical state in a system with constraints in the displacement field are the same as in systems without constraints whenever there is only one control parameter. The topic of imperfection sensitivity has not been addressed in this paper, but it is seen as an important topic for future research.

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